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ORIGINAL PAPER

A p -adic analogue of Siegel's theorem on sums of squares

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Abstract

Siegel proved that every totally positive element of a number field K is the sum of four squares, so in particular the Pythagoras number is uniformly bounded across number fields. The p -adic Kochen operator provides a p -adic analogue of squaring, and a certain localisation of the ring generated by this operator consists of precisely the totally p -integral elements of K . We use this to formulate and prove a p -adic analogue of Siegel's theorem, by introducing the p -Pythagoras number of a general field, and showing that this number is uniformly bounded across number fields. We also generally study fields with finite p -Pythagoras number and show that the growth of the p -Pythagoras number in finite extensions is bounded.

KEYWORDS

Kochen operator, number fields, p -valuations

MSC (2010)

11E25, 11S99, 11U09, 12D15

1 | INTRODUCTION

The study of sums of squares has a long history. In the context of the integers, Fermat, Euler, Lagrange and many others studied which integers are a sum of a certain number of square integers. The possibly most famous result in this direction is Lagrange's Four Squares Theorem [13, Thm. 369] that every nonnegative integer is the sum of four squares. In fact, earlier Euler had proved a version of this theorem for \mathbb{Q} : every nonnegative rational number is the sum of four square rational numbers. A comprehensive history of these theorems may be found in [6, Chapter VIII]. In the other direction, for both \mathbb{Z} and \mathbb{Q} there exist nonnegative numbers that cannot be written as a sum of three squares. The *Pythagoras number* $\pi(F)$ of a field F is the smallest n such that

$$\{x_1^2 + \cdots + x_m^2 \mid x_1, \dots, x_m \in F, m \in \mathbb{N}\} = \{x_1^2 + \cdots + x_n^2 \mid x_1, \dots, x_n \in F\}.$$

Using this terminology, Euler's theorem becomes the statement that $\pi(\mathbb{Q}) = 4$. The following generalization of Euler's theorem was conjectured by Hilbert and proven by Siegel in [25], cf. [20, Ch. 7, §1, 1.4]:

Theorem 1.1 (Siegel). *For all number fields F , $\pi(F) \leq 4$.*

The study of the Pythagoras number of a field is intimately related to the study of the orderings on that field, since by a theorem of Artin and Schreier the sums of squares are precisely the totally positive elements. In a number field F , these can be described simply as those elements that are mapped to $\mathbb{R}_{\geq 0}$ by every embedding of F into \mathbb{R} , cf. [20, Ch. 3 and 7].

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We define and study a p -adic version of the Pythagoras number, namely the p -Pythagoras number $\pi_p(F)$ of a field F , or more generally the (\mathfrak{p}, τ) -Pythagoras number, see Section 2.2 for the definition. Just like the Pythagoras number gives information on the set of totally positive elements, the p -Pythagoras number relates to the set of totally p -integral elements, which in a number field F can be described simply as those elements that are mapped to \mathbb{Z}_p by every embedding of F into \mathbb{Q}_p . Our main result is an inexplicit analogue of Siegel's theorem:

Theorem 1.2. *Let p be a prime number. There exists $N_p \in \mathbb{N}$ such that $\pi_p(F) \leq N_p$ for every number field F .*

This result will be deduced from the more general Theorem 4.9. We also give some general results on fields F with finite (\mathfrak{p}, τ) -Pythagoras number and prove in Theorem 5.9 that the growth of the (\mathfrak{p}, τ) -Pythagoras number is bounded in finite extensions. As an application, we show in Corollary 6.5 that for every open-closed subset of the p -adic spectrum of F , the associated holomorphy ring is diophantine. A further application can be found in the forthcoming work [2], in which we use the results of this paper to show that rings of formal power series over number fields are \mathbb{Z} -diophantine in their quotient fields.

2 | THE (\mathfrak{p}, τ) -PYTHAGORAS NUMBER

2.1 | p -valuations

A (Krull) valuation v on a field F is a p -valuation if it has a finite residue field \bar{F}_v of characteristic p and value group $v(F^\times)$ such that the interval $(0, v(p)]$ is finite. A (finite) prime \mathfrak{P} of a field F is an equivalence class of p -valuations on F (for the usual notion of equivalence of valuations), for some prime number p . We write $v_{\mathfrak{P}}$ for a representative of \mathfrak{P} which has \mathbb{Z} as smallest non-trivial convex subgroup of the value group. See [22] for basics regarding p -valuations, and [10] for details on this notion of prime and some of the following definitions.

Example 2.1. The primes of a number field K correspond precisely to the finite places in the usual sense and we will identify them. If $K = \mathbb{Q}$ and p is a prime number then v_p denotes the usual p -adic valuation, and we denote the corresponding prime also by p .

For the rest of this work we fix a triple (K, \mathfrak{p}, τ) , where K is a number field, \mathfrak{p} is a finite prime of K , and τ is a pair of natural numbers $(e, f) \in \mathbb{N}^2$. We denote by $t_{\mathfrak{p}}$ a uniformizer of $v_{\mathfrak{p}}$, i.e. an element with $v_{\mathfrak{p}}(t_{\mathfrak{p}}) = 1$, we let q denote the size of the residue field $\bar{K}_{v_{\mathfrak{p}}}$.

For a field extension F/K with \mathfrak{P} a prime of F lying above \mathfrak{p} , the *relative initial ramification* is $e(\mathfrak{P}|\mathfrak{p}) := v_{\mathfrak{P}}(t_{\mathfrak{p}})$, the *relative residue degree* is $f(\mathfrak{P}|\mathfrak{p}) := [\bar{F}_{v_{\mathfrak{P}}} : \bar{K}_{v_{\mathfrak{p}}}]$, and the pair $(e(\mathfrak{P}|\mathfrak{p}), f(\mathfrak{P}|\mathfrak{p}))$ is the *relative type* of \mathfrak{P} over \mathfrak{p} . We say \mathfrak{P} is of relative type at most τ if $e(\mathfrak{P}|\mathfrak{p})$ is no greater than e , and $f(\mathfrak{P}|\mathfrak{p})$ divides f . Likewise, for $\tau' = (e', f')$ we write $\tau \leq \tau'$ if $e \leq e'$ and $f \mid f'$. We denote by $S(F)$ the set of primes of F , by $S_{\mathfrak{p}}^*(F) \subseteq S(F)$ the set of those primes \mathfrak{P} of F lying above \mathfrak{p} , and by $S_{\mathfrak{p}}^{\tau}(F) \subseteq S_{\mathfrak{p}}^*(F)$ the subset of those primes \mathfrak{P} of F which are of relative type at most τ over \mathfrak{p} . The corresponding *holomorphy ring* is

$$R_{\mathfrak{p}}^{\tau}(F) := \bigcap_{\mathfrak{P} \in S_{\mathfrak{p}}^{\tau}(F)} \mathcal{O}_{\mathfrak{P}},$$

where $\mathcal{O}_{\mathfrak{P}}$ is the valuation ring of \mathfrak{P} , and

$$\Gamma_{\mathfrak{p}}^{\tau}(F) := \left\{ \frac{a}{1 + t_{\mathfrak{p}} b} \mid a, b \in \mathcal{O}_{\mathfrak{p}} \left[\gamma_{\mathfrak{p}, t_{\mathfrak{p}}}^{\tau}(F) \right], 1 + t_{\mathfrak{p}} b \neq 0 \right\}$$

is the corresponding *Kochen ring*, where

$$\gamma_{\mathfrak{p}, t_{\mathfrak{p}}}^{\tau}(X) := \frac{1}{t_{\mathfrak{p}}} \cdot \left(\frac{X^{q^f} - X}{(X^{q^f} - X)^2 - 1} \right)^e$$

is the *Kochen operator*. Here and in what follows, if $\gamma \in F(X)$ is a rational function, we mean by $\gamma(F)$ the image of γ on $F \setminus \{\text{poles of } \gamma\}$. Note that $\Gamma_{\mathfrak{p}}^{\tau}(F)$ does not depend on the choice of $t_{\mathfrak{p}}$, since the quotient of two uniformizers of $v_{\mathfrak{p}}$ is an element of $\mathcal{O}_{\mathfrak{p}}^{\times}$. Recall that $R_{\mathfrak{p}}^{\tau}(F)$ is the integral closure of $\Gamma_{\mathfrak{p}}^{\tau}(F)$, with equality in the case $e = 1$, see [22, Cor. 6.9] and the subsequent discussion for more details.

Example 2.2. If \mathfrak{p} is any place of the number field K , we denote by $K_{\mathfrak{p}}$ the completion of K with respect to \mathfrak{p} . If \mathfrak{p} is a finite place, then $K_{\mathfrak{p}}$ is a non-archimedean local field and \mathfrak{p} extends to a unique prime \mathfrak{P} of $K_{\mathfrak{p}}$ of the same type, so $R_{\mathfrak{p}}^{\tau}(K_{\mathfrak{p}}) = R_{\mathfrak{p}}^{(1,1)}(K_{\mathfrak{p}}) = \mathcal{O}_{\mathfrak{P}}$. In fact, any non-archimedean local field E of characteristic zero carries a unique prime, whose valuation ring we denote by \mathcal{O}_E , cf. [22, Thm. 6.15]. We say that an extension of non-archimedean local fields is of relative type at most τ if this is true for the respective primes.

The real holomorphy ring of F is the intersection of the positive cones of the orderings on F , i.e. the set of elements that are nonnegative under every ordering on F . By the theorem of Artin and Schreier it can alternatively be described as the set of sums of squares, and the classical Pythagoras number may be seen as a measure of the complexity of this description in terms of squares. The holomorphy ring $R_{\mathfrak{p}}^{\tau}(F)$ is defined above as an intersection of the valuation rings of certain p -valuations, and it also equals the integral closure of $\Gamma_{\mathfrak{p}}^{\tau}(F)$. Thus a p -adic analogue of the Pythagoras number should somehow measure the complexity of the description of $R_{\mathfrak{p}}^{\tau}(F)$ in terms of the rational function $\gamma_{\mathfrak{p},t_{\mathfrak{p}}}^{\tau}$. We now define such a p -adic analogue.

2.2 | The (\mathfrak{p}, τ) -Pythagoras number

Let F/K be an extension. For $g \in \mathcal{O}_{\mathfrak{p}}[X_1, \dots, X_n]$, we write

$$R_{\mathfrak{p},g,t_{\mathfrak{p}}}^{\tau}(F) := \left\{ \frac{a}{1+t_{\mathfrak{p}}b} \mid a, b \in g(\gamma_{\mathfrak{p},t_{\mathfrak{p}}}^{\tau}(F), \dots, \gamma_{\mathfrak{p},t_{\mathfrak{p}}}^{\tau}(F)), 1+t_{\mathfrak{p}}b \neq 0 \right\},$$

and for $n \geq 1$

$$R_{\mathfrak{p},g,t_{\mathfrak{p}},n}^{\tau}(F) := \left\{ x \in F \mid x^m + a_{m-1}x^{m-1} + \dots + a_0 = 0 \text{ with } 1 \leq m \leq n, a_0, \dots, a_{m-1} \in R_{\mathfrak{p},g,t_{\mathfrak{p}}}^{\tau}(F) \right\}.$$

We denote by $\mathcal{P}_{\mathfrak{p},n}$ the finite set of those $g \in \mathcal{O}_{\mathfrak{p}}[X_1, \dots, X_n]$ of degree and height at most n (cf. [4, Def. 1.6.1]). We write

$$R_{\mathfrak{p},n}^{\tau}(F) := \bigcup_{t_{\mathfrak{p}}} \bigcup_{g \in \mathcal{P}_{\mathfrak{p},n}} R_{\mathfrak{p},g,t_{\mathfrak{p}},n}^{\tau}(F),$$

where $t_{\mathfrak{p}}$ varies over those (finitely many) elements of the ring of integers \mathcal{O}_K which are uniformizers for \mathfrak{p} of minimal height. Then $(R_{\mathfrak{p},n}^{\tau}(F))_{n \in \mathbb{N}}$ is an increasing chain of subsets of F and

$$R_{\mathfrak{p}}^{\tau}(F) = \bigcup_{n \in \mathbb{N}} R_{\mathfrak{p},n}^{\tau}(F).$$

The (\mathfrak{p}, τ) -Pythagoras number $\pi_{\mathfrak{p}}^{\tau}(F)$ of F is the smallest n such that

$$R_{\mathfrak{p}}^{\tau}(F) = R_{\mathfrak{p},n}^{\tau}(F),$$

and we write $\pi_{\mathfrak{p}}^{\tau}(F) = \infty$ if there is no such n . In other words,

$$\pi_{\mathfrak{p}}^{\tau}(F) := \inf \left\{ n \in \mathbb{N} \mid R_{\mathfrak{p}}^{\tau}(F) = R_{\mathfrak{p},n}^{\tau}(F) \right\} \in \mathbb{N} \cup \{\infty\}.$$

In the case $K = \mathbb{Q}$, $\mathfrak{p} = p$ and $\tau = (1, 1)$, we write $R_p(F)$ and $\pi_p(F)$, omitting the relative type $(1, 1)$, and we speak of the p -Pythagoras number. We also write $\gamma_p := \gamma_{p,p}^{(1,1)}$, and note that the only two uniformizers (of the prime p) in \mathbb{Z} of minimal height are p and $-p$, with $\gamma_{p,-p}^{(1,1)} = -\gamma_p$. We discuss some possible variations of our definition of the (\mathfrak{p}, τ) -Pythagoras number in Remarks 3.11 and 3.12.

Example 2.3. Since \mathbb{C} is algebraically closed and carries no p -valuation, we have

$$R_p(\mathbb{C}) = \mathbb{C} = \gamma_p(\mathbb{C}),$$

in particular $\pi_p(\mathbb{C}) = 1$.

Example 2.4. It follows easily from Hensel's lemma that

$$R_p(\mathbb{Q}_p) = \mathbb{Z}_p = \gamma_p(\mathbb{Q}_p),$$

in particular $\pi_p(\mathbb{Q}_p) = 1$, see [22, Thm. 6.15].

Example 2.5. In [11, Lem. 3.02] it is shown that every so-called pseudo p -adically closed field F (where pseudo p -adically closed means that a certain geometric local-global principle holds for varieties over F) satisfies

$$R_p(F) = \gamma_p(F) + \gamma_p(F) + \gamma_p(F),$$

hence $\pi_p(F) \leq 3$. This applies for example to the field \mathbb{Q}^{lp} of totally p -adic algebraic numbers by a result of Moret–Bailly [17], where the local-global principle takes the following simple form: If V is a geometrically irreducible smooth variety over \mathbb{Q}^{lp} which has a \mathbb{Q}_p -rational point for every embedding of \mathbb{Q}^{lp} into \mathbb{Q}_p , then it has a \mathbb{Q}^{lp} -rational point.

It is known that there are fields F with $\pi(F) = \infty$, for example $F = \mathbb{R}(x_1, x_2, \dots)$, see [15, Ch. XI, Example 5.9(5)]. On the other hand, we do not know if $\pi_p(F) = \infty$ for any field:

Question 2.6. Is $\pi_p(\mathbb{Q}(X_1, X_2, \dots)) = \infty$?

2.3 | Explicit bounds and uniformity in p

We now prove a few rather elementary statements about $\pi_p(\mathbb{Q})$. We will drop the relative type $\tau = (1, 1)$ from all notation. Let ℓ be a prime number distinct from p .

Lemma 2.7. We have $\gamma_p(\mathbb{Q}) \subseteq \mathbb{Z}_{(\ell)}$ if and only if neither $X^p - X + 1$ nor $X^p - X - 1$ has a zero in \mathbb{F}_ℓ .

Proof. Let $x \in \mathbb{Q}$, recall that $\gamma_p(x) = \frac{1}{p}((x^p - x) - (x^p - x)^{-1})^{-1}$ and denote by v_ℓ the ℓ -adic valuation. If $v_\ell(x^p - x) < 0$ or $v_\ell(x^p - x) > 0$, then $v_\ell(\gamma_p(x)) > 0$. If $v_\ell(x^p - x) = 0$, then $x \in \mathbb{Z}_{(\ell)}$, and $v_\ell(\gamma_p(x)) < 0$ if and only if $(x^p - x) - (x^p - x)^{-1} \equiv 0 \pmod{\ell}$, which means that $x^p - x \equiv \pm 1 \pmod{\ell}$. \square

Proposition 2.8. $\mathbb{Z}[\gamma_p(\mathbb{Q})] \not\subseteq \mathbb{Z}_{(p)}$.

Proof. There exists a prime number $\ell \neq p$ such that $\mathbb{Z}[\gamma_p(\mathbb{Q})]$ is contained in $\mathbb{Z}_{(\ell)}$ by Lemma 2.7: specifically, the criterion given there is satisfied by $\ell = 2$ if p is odd and by $\ell = 17$ for $p = 2$. \square

Lemma 2.9. If $\ell - 1 \mid p - 1$ then $\gamma_p(\mathbb{Q}) \subseteq \ell \mathbb{Z}_{(\ell)}$.

Proof. If $\ell - 1 \mid p - 1$, then $x^p - x = 0$ for all $x \in \mathbb{F}_\ell$. Thus $v_\ell(\gamma_p(x)) > 0$ for all $x \in \mathbb{Q}$, where v_ℓ is the ℓ -adic valuation. \square

Proposition 2.10. For every finite set $\mathcal{P} \subseteq \mathbb{Q}[X_1, X_2, \dots]$, there exist some p and $\ell \neq p$ with

$$\bigcup_{g \in \mathcal{P}} R_{p,g,p}(\mathbb{Q}) \subseteq \mathbb{Z}_{(\ell)}.$$

In particular, $\sup_p \pi_p(\mathbb{Q}) = \infty$.

Proof. Choose $\ell > |\mathcal{P}| + 1$ such that $\mathcal{P} \subseteq \mathbb{Z}_{(\ell)}[X_1, X_2, \dots]$. There exists $a \in \mathbb{Z}$ such that $a \not\equiv 0 \pmod{\ell}$ and $a \not\equiv g(0, \dots, 0) \pmod{\ell}$ for every $g \in \mathcal{P}$. By Dirichlet's theorem on primes in arithmetic progressions (see [18, VII, (13.2)]), there exist infinitely many primes $p > \ell$ with $p \equiv 1 \pmod{\ell - 1}$ and $p \equiv -a^{-1} \pmod{\ell}$. Then

$$g(\gamma_p(\mathbb{Q}), \dots, \gamma_p(\mathbb{Q})) \subseteq g(0, \dots, 0) + \ell \mathbb{Z}_{(\ell)}$$

by Lemma 2.9, hence $1 + pg(\gamma_p(\mathbb{Q}), \dots, \gamma_p(\mathbb{Q})) \subseteq \mathbb{Z}_{(\ell)}^\times$ by the choice of a and p . Thus $R_{p,g,p}(\mathbb{Q}) \subseteq \mathbb{Z}_{(\ell)}$ for every $g \in \mathcal{P}$.

By the integral closedness of $\mathbb{Z}_{(\ell)}$ this implies $R_{p,g,p,n}(\mathbb{Q}) \subseteq \mathbb{Z}_{(\ell)}$ for every n . Note that $R_{p,g,-p,n}(F) = -R_{p,g^*,p,n}(F)$, where $g^*(X_1, \dots, X_n) = -g(-X_1, \dots, -X_n)$ has the same height as g . Therefore, applying the above to the set \mathcal{P} of all $f \in \mathbb{Q}[X_1, \dots, X_n]$ of degree and height at most n , we obtain ℓ and $p > \ell$ with

$$\bigcup_{g \in \mathcal{P}_{p,n}} (R_{p,g,p,n}(F) \cup R_{p,g,-p,n}(F)) \subseteq \bigcup_{p \in \mathcal{P}} R_{p,g,p,n}(F) \subseteq \mathbb{Z}_{(\ell)},$$

and therefore $\pi_p(\mathbb{Q}) > n$. \square

2.4 | The Kochen operator

For later use, we explore several simple properties of the Kochen operator. Let F/K be any extension.

Lemma 2.11. Let $\mathfrak{P} \in S_{\mathfrak{p}}^*(F)$ and suppose that $x \in F$ is not a pole of $\gamma_{\mathfrak{p}, t_{\mathfrak{p}}}^{\tau}$. Then

$$v_{\mathfrak{P}}(\gamma_{\mathfrak{p}, t_{\mathfrak{p}}}^{\tau}(x)) = \begin{cases} -eq^f v_{\mathfrak{P}}(x) - v_{\mathfrak{P}}(t_{\mathfrak{p}}) & \text{if } v_{\mathfrak{P}}(x) < 0, \\ ev_{\mathfrak{P}}(x) - v_{\mathfrak{P}}(t_{\mathfrak{p}}) & \text{if } v_{\mathfrak{P}}(x) > 0, \\ ev_{\mathfrak{P}}(x^{q^f} - x) - v_{\mathfrak{P}}(t_{\mathfrak{p}}) & \text{if } v_{\mathfrak{P}}(x) = 0 \text{ and } v_{\mathfrak{P}}(x^{q^f} - x) > 0, \\ -ev_{\mathfrak{P}}((x^{q^f} - x)^2 - 1) - v_{\mathfrak{P}}(t_{\mathfrak{p}}) & \text{if } v_{\mathfrak{P}}(x) = 0 \text{ and } v_{\mathfrak{P}}(x^{q^f} - x) = 0. \end{cases}$$

Proof. This is a matter of calculating valuations. □

Lemma 2.12. Let $\mathfrak{P} \in S_{\mathfrak{p}}^*(F)$. Suppose that $x \in F$ is not a pole of $\gamma_{\mathfrak{p}, t_{\mathfrak{p}}}^{\tau}$ and satisfies either

- (i) $0 < (e+1)v_{\mathfrak{P}}(x) \leq v_{\mathfrak{P}}(t_{\mathfrak{p}})$, or
- (ii) $v_{\mathfrak{P}}(x) = 0$ and $[\mathbb{F}_q(\text{res}_{\mathfrak{P}}(x)) : \mathbb{F}_q] \nmid f$, where $\text{res}_{\mathfrak{P}}(x)$ is the residue of x .

Then

$$v_{\mathfrak{P}}(\gamma_{\mathfrak{p}, t_{\mathfrak{p}}}^{\tau}(x)) \leq -\frac{1}{e+1}v_{\mathfrak{P}}(t_{\mathfrak{p}}) < 0.$$

Proof. In case (i), Lemma 2.11 gives that

$$v_{\mathfrak{P}}(\gamma_{\mathfrak{p}, t_{\mathfrak{p}}}^{\tau}(x)) = ev_{\mathfrak{P}}(x) - v_{\mathfrak{P}}(t_{\mathfrak{p}}) \leq -\frac{1}{e+1}v_{\mathfrak{P}}(t_{\mathfrak{p}}).$$

In case (ii), the residue of x is not a root of $X^{q^f} - X$, and so

$$v_{\mathfrak{P}}(\gamma_{\mathfrak{p}, t_{\mathfrak{p}}}^{\tau}(x)) = -ev_{\mathfrak{P}}((x^{q^f} - x)^2 - 1) - v_{\mathfrak{P}}(t_{\mathfrak{p}}) \leq -v_{\mathfrak{P}}(t_{\mathfrak{p}}) \leq -\frac{1}{e+1}v_{\mathfrak{P}}(t_{\mathfrak{p}}),$$

also by Lemma 2.11. □

Lemma 2.13. Let $\mathfrak{P} \in S_{\mathfrak{p}}^*(F)$, let $x, y \in F$, and suppose that x is not a pole of $\gamma_{\mathfrak{p}, t_{\mathfrak{p}}}^{\tau}$, and $v_{\mathfrak{P}}(\gamma_{\mathfrak{p}, t_{\mathfrak{p}}}^{\tau}(x)) < 0$. If $v_{\mathfrak{P}}(x - y) \geq v_{\mathfrak{P}}(t_{\mathfrak{p}})$, then also y is not a pole of $\gamma_{\mathfrak{p}, t_{\mathfrak{p}}}^{\tau}$, and $v_{\mathfrak{P}}(\gamma_{\mathfrak{p}, t_{\mathfrak{p}}}^{\tau}(y)) < 0$.

Proof. If $v_{\mathfrak{P}}(x) \leq 0$, then in particular $v_{\mathfrak{P}}(x) < v_{\mathfrak{P}}(t_{\mathfrak{p}})$, while if $v_{\mathfrak{P}}(x) > 0$, then $v_{\mathfrak{P}}(\gamma_{\mathfrak{p}, t_{\mathfrak{p}}}^{\tau}(x)) = ev_{\mathfrak{P}}(x) - v_{\mathfrak{P}}(t_{\mathfrak{p}})$ by Lemma 2.11, hence $v_{\mathfrak{P}}(\gamma_{\mathfrak{p}, t_{\mathfrak{p}}}^{\tau}(x)) < 0$ implies that $v_{\mathfrak{P}}(x) < v_{\mathfrak{P}}(t_{\mathfrak{p}})$ also in this case. Therefore, in either case we conclude from $v_{\mathfrak{P}}(x - y) \geq v_{\mathfrak{P}}(t_{\mathfrak{p}})$ that $v_{\mathfrak{P}}(x) = v_{\mathfrak{P}}(y)$. We make a case distinction:

Suppose first that $v_{\mathfrak{P}}(x) \neq 0$. By Lemma 2.11, in this case, $v_{\mathfrak{P}}(\gamma_{\mathfrak{p}, t_{\mathfrak{p}}}^{\tau}(x))$ depends only on $v_{\mathfrak{P}}(x)$. Therefore $v_{\mathfrak{P}}(\gamma_{\mathfrak{p}, t_{\mathfrak{p}}}^{\tau}(y)) = v_{\mathfrak{P}}(\gamma_{\mathfrak{p}, t_{\mathfrak{p}}}^{\tau}(x)) < 0$.

Suppose now that $v_{\mathfrak{P}}(x) = 0$. As $x - y$ divides $x^{q^f} - y^{q^f}$ in $\mathcal{O}_{\mathfrak{P}}$, we have that $v_{\mathfrak{P}}(y^{q^f} - y - x^{q^f} + x) \geq v_{\mathfrak{P}}(x - y) \geq v_{\mathfrak{P}}(t_{\mathfrak{p}})$. If $v_{\mathfrak{P}}(x^{q^f} - x) = 0$, then in particular $v_{\mathfrak{P}}(x^{q^f} - x) < v_{\mathfrak{P}}(t_{\mathfrak{p}})$, while if $v_{\mathfrak{P}}(x^{q^f} - x) > 0$, then $v_{\mathfrak{P}}(\gamma_{\mathfrak{p}, t_{\mathfrak{p}}}^{\tau}(x)) < 0$ implies that $v_{\mathfrak{P}}(x^{q^f} - x) < \frac{1}{e}v_{\mathfrak{P}}(t_{\mathfrak{p}}) \leq v_{\mathfrak{P}}(t_{\mathfrak{p}})$ by Lemma 2.11. Thus $v_{\mathfrak{P}}(y^{q^f} - y) = v_{\mathfrak{P}}(x^{q^f} - x)$ in both cases. If $v_{\mathfrak{P}}(x^{q^f} - x) = 0$, then Lemma 2.11 gives immediately that $v_{\mathfrak{P}}(\gamma_{\mathfrak{p}, t_{\mathfrak{p}}}^{\tau}(y)) < 0$, while if $v_{\mathfrak{P}}(x^{q^f} - x) > 0$, then Lemma 2.11 shows that $v_{\mathfrak{P}}(\gamma_{\mathfrak{p}, t_{\mathfrak{p}}}^{\tau}(x))$ depends only on $v_{\mathfrak{P}}(x^{q^f} - x)$, hence $v_{\mathfrak{P}}(\gamma_{\mathfrak{p}, t_{\mathfrak{p}}}^{\tau}(y)) = v_{\mathfrak{P}}(\gamma_{\mathfrak{p}, t_{\mathfrak{p}}}^{\tau}(x)) < 0$. □

3 | DIOPHANTINE FAMILIES

A *diophantine* subset of a field F is the image of the F -rational points of some F -variety V under a morphism $V \rightarrow \mathbb{A}_F^1$. As we want to discuss questions of uniformity we use the following slightly more sophisticated notion: An *n -dimensional diophantine*

family over K is a map D from the class of field extensions F of K to sets which is given by finitely many polynomials $f_1, \dots, f_r \in K[X_1, \dots, X_n, Y_1, \dots, Y_m]$, for some m , in the sense that

$$D(F) = \{x \in F^n \mid \exists y \in F^m : f_1(x, y) = 0, \dots, f_r(x, y) = 0\}$$

for every extension F/K . In this case, we say that the polynomials f_1, \dots, f_r define D . Note that if E/F is an extension, then $D(F) \subseteq D(E)$.

Remark 3.1. From the point of view of algebraic geometry, an n -dimensional diophantine family D over K is given by a morphism of (not necessarily irreducible) K -varieties $\varphi : V \rightarrow \mathbb{A}_K^n$ in the sense that $D(F) = \varphi(V(F))$ for every extension F/K .

Remark 3.2. From the point of view of model theory, an n -dimensional diophantine family D over K is given by an existential formula $\varphi(x_1, \dots, x_n)$ in the language of rings with free variables among x_1, \dots, x_n and parameters from K , in the sense that for every extension F/K , $D(F)$ is the set defined by φ in F , i.e. the set of $a \in F^n$ such that $F \models \varphi(a)$. Such a formula is equivalent (modulo the theory of fields) to a formula of the form

$$\exists y_1 \dots y_m : \bigwedge_{i=1}^r f_i(x_1, \dots, x_n, y_1, \dots, y_m) = 0$$

with $f_1, \dots, f_r \in K[X_1, \dots, X_n, Y_1, \dots, Y_m]$.

Most of the usual constructions for diophantine sets (see e.g. [24]) go through for diophantine families:

Lemma 3.3. *If D_1, D_2 are n -dimensional diophantine families over K , then there are n -dimensional diophantine families $D_1 \cup D_2$ and $D_1 \cap D_2$ over K such that $(D_1 \cup D_2)(F) = D_1(F) \cup D_2(F)$ and $(D_1 \cap D_2)(F) = D_1(F) \cap D_2(F)$ for every F/K .*

Proof. Suppose that the polynomials $f_1, \dots, f_r \in K[X_1, \dots, X_n, Y_1, \dots, Y_m]$ define D_1 and that the polynomials $g_1, \dots, g_s \in K[X_1, \dots, X_n, Z_1, \dots, Z_l]$ define D_2 . We may assume that the variables Y_i and Z_j are distinct. We observe that $f_1, \dots, f_r, g_1, \dots, g_s$ define $D_1 \cap D_2$. Slightly less trivially, we have that $f_1 g_1, \dots, f_r g_s$ define $D_1 \cup D_2$. \square

Lemma 3.4. *Suppose that D_1 and D_2 are n_1 - respectively n_2 -dimensional diophantine families over K . Then there is an $(n_1 + n_2)$ -dimensional diophantine family $D_1 \times D_2$ over K such that $(D_1 \times D_2)(F) = D_1(F) \times D_2(F)$ for every F/K .*

Proof. Suppose that the polynomials $f_1, \dots, f_r \in K[X_1, \dots, X_{n_1}, Y_1, \dots, Y_m]$ define D_1 and that the polynomials $g_1, \dots, g_s \in K[X'_1, \dots, X'_{n_2}, Z_1, \dots, Z_l]$ define D_2 . This time, we suppose that all the variables X_i, X'_i, Y_i, Z_i are distinct. Then the polynomials $f_1, \dots, f_r, g_1, \dots, g_s$ define $D_1 \times D_2$. \square

Lemma 3.5. *Let D be an n -dimensional diophantine family over K and let $f = \left(\frac{g_1}{h_1}, \dots, \frac{g_k}{h_k}\right)$ be a tuple of rational functions with $g_i, h_i \in K[X_1, \dots, X_n]$ such that for every i the polynomials g_i and h_i are coprime. Then there is an k -dimensional diophantine family fD with*

$$(fD)(F) = \left\{ \left(\frac{g_1(x)}{h_1(x)}, \dots, \frac{g_k(x)}{h_k(x)} \right) \mid x \in D(F), h_i(x) \neq 0 \text{ for all } i \right\}$$

for every F/K .

Proof. Let $f_1, \dots, f_r \in K[X_1, \dots, X_n, Y_1, \dots, Y_m]$ define D . Then a tuple $(z_1, \dots, z_k) \in F^k$ is an element of the right hand side if and only if there exists $(x_1, \dots, x_n, y_1, \dots, y_m, w_1, \dots, w_k) \in F^{n+m+k}$ such that

1. $g_i(x_1, \dots, x_n) - z_i h_i(x_1, \dots, x_n) = 0$ for all $i = 1, \dots, k$,
2. $w_i h_i(x_1, \dots, x_n) = 1$ for all $i = 1, \dots, k$, and
3. $f_j(x_1, \dots, x_n, y_1, \dots, y_m) = 0$ for all $j = 1, \dots, r$.

Each of these conditions is the vanishing of a polynomial in the variables $W_1, \dots, W_k, X_1, \dots, X_n, Y_1, \dots, Y_m$ and Z_1, \dots, Z_k over K . \square

Remark 3.6. Perhaps the most trivial 1-dimensional diophantine family over K is the one assigning the set F to every field F/K . As described above in Section 2.1, given a rational function $\gamma \in K(X)$ and a field F/K , we write $\gamma(F)$ to mean the image under

γ of $F \setminus \{\text{poles of } \gamma\}$. By this small abuse of notation, γ may be identified with the map which sends a field F/K to its image $\gamma(F)$ under γ . Then by Lemma 3.5, γ is a 1-dimensional diophantine family over K . This applies in particular to the Kochen operator $\gamma_{\mathfrak{p}, t_{\mathfrak{p}}}^{\tau}$.

Lemma 3.7. *If D is an n -dimensional diophantine family over K and $a = (a_1, \dots, a_r) \in K^r$, $r < n$, then there is an $(n-r)$ -dimensional family D_a over K with*

$$D_a(F) = \{x \in F^{n-r} \mid (x, a) \in D(F)\}$$

for every F/K .

Proof. Again, let $f_1, \dots, f_r \in K[X_1, \dots, X_n, Y_1, \dots, Y_m]$ define D . We write

$$g_i(X_1, \dots, X_{n-r}, Y_1, \dots, Y_m) := f_i(X_1, \dots, X_{n-r}, a_1, \dots, a_r, Y_1, \dots, Y_m).$$

Then the polynomials $g_1, \dots, g_r \in K[X_1, \dots, X_{n-r}, Y_1, \dots, Y_m]$ define the $(n-r)$ -dimensional diophantine family D_a over K . \square

Example 3.8. Each of the $R_{\mathfrak{p}, n}^{\tau}$ is a 1-dimensional diophantine family over K .

Proposition 3.9. *Let D, D_1, D_2, \dots be n -dimensional diophantine families over K . If $D(F) \subseteq \bigcup_{i \in \mathbb{N}} D_i(F)$ for every extension F/K , then there exists N such that $D(F) \subseteq \bigcup_{i=1}^N D_i(F)$ for every extension F/K .*

Proof. In light of Remark 3.2, this is a direct consequence of the compactness theorem of model theory, see for example [16, Thm. 2.1.4]. \square

Proposition 3.10. *Let D be a 1-dimensional diophantine family over K and let \mathcal{K} be a class of extensions of K . If*

- (i) $D(L) = R_{\mathfrak{p}}^{\tau}(L)$ for every $L \in \mathcal{K}$, and
- (ii) $D(E) \subseteq \mathcal{O}_E$ for every finite extension $E/K_{\mathfrak{p}}$ of relative type at most τ ,

then there exists N such that $\pi_{\mathfrak{p}}^{\tau}(L) \leq N$ for every $L \in \mathcal{K}$.

Proof. Let F be any extension of K . For $\mathfrak{P} \in S_{\mathfrak{p}}^{\tau}(F)$ let (F', \mathfrak{P}') denote a p -adic closure of (F, \mathfrak{P}) (see [22, §3]). By the p -adic Lefschetz principle, the assumption (ii) implies that $D(F') \subseteq \mathcal{O}_{\mathfrak{P}'}$, in particular $D(F) \subseteq \mathcal{O}_{\mathfrak{P}'} \cap F = \mathcal{O}_{\mathfrak{P}}$. (In model-theoretic terms, F' is elementarily equivalent, in the language of valued fields, to a finite extension E of $K_{\mathfrak{p}}$ of relative type at most τ . More precisely, if F_0 denotes the algebraic part of F' , then both $F_0 K_{\mathfrak{p}}$ and F' are elementary extensions of F_0 by [22, Thm. 5.1].) In particular, $D(F) \subseteq \bigcap_{\mathfrak{P} \in S_{\mathfrak{p}}^{\tau}(F)} \mathcal{O}_{\mathfrak{P}} = R_{\mathfrak{p}}^{\tau}(F)$. So since $R_{\mathfrak{p}}^{\tau}(F) = \bigcup_{n=1}^{\infty} R_{\mathfrak{p}, n}^{\tau}(F)$, by Proposition 3.9 there exists N such that $D(F) \subseteq \bigcup_{n=1}^N R_{\mathfrak{p}, n}^{\tau}(F)$ for every F/K . In fact $(R_{\mathfrak{p}, n}^{\tau}(F))_{n \in \mathbb{N}}$ is an increasing chain, so $D(F) \subseteq R_{\mathfrak{p}, N}^{\tau}(F)$. Thus for $L \in \mathcal{K}$, (i) implies that $R_{\mathfrak{p}}^{\tau}(L) = D(L) \subseteq R_{\mathfrak{p}, N}^{\tau}(L)$, which shows that $\pi_{\mathfrak{p}}^{\tau}(L) \leq N$. \square

Remark 3.11. We also have the following converse: If $\pi_{\mathfrak{p}}^{\tau}(L) \leq N$ for all $L \in \mathcal{K}$, then $D = R_{\mathfrak{p}, N}^{\tau}$ is a diophantine family satisfying both conditions. This indicates that while our definition of $\pi_{\mathfrak{p}}^{\tau}$ depends on the construction of the height function on polynomials over $\mathcal{O}_{\mathfrak{p}}$, the property of a class \mathcal{K} to have bounded (\mathfrak{p}, τ) -Pythagoras number is a very robust notion and does not depend on the details of the height function.

Remark 3.12. The notion that a class \mathcal{K} has bounded (\mathfrak{p}, τ) -Pythagoras number is robust in a further sense: under taking a suitable alternative for the Kochen operator. Consider a rational function $\delta \in K(X)$ and suppose that $R_{\mathfrak{p}}^{\tau}(F)$ is the integral closure in F of the ring

$$R'(F) := \left\{ \frac{a}{1 + t_{\mathfrak{p}} b} \mid a, b \in \mathcal{O}_{\mathfrak{p}}[\delta(F)], 1 + t_{\mathfrak{p}} b \neq 0 \right\},$$

for every extension F/K . We introduce a new 1-dimensional diophantine family R'_n over K , by defining $R'_n(F)$ in terms of δ exactly as $R_{\mathfrak{p}, n}^{\tau}(F)$ is defined in terms of $\gamma_{\mathfrak{p}, t_{\mathfrak{p}}}^{\tau}$. Then

$$R_{\mathfrak{p}}^{\tau}(F) = \bigcup_{n=1}^{\infty} R'_n(F),$$

for all F/K . Simply adapting the proof of Proposition 3.10, a class \mathcal{K} of extensions of K has bounded (\mathfrak{p}, τ) -Pythagoras number if and only if there is $M \in \mathbb{N}$ such that $R'_M(L) = R_{\mathfrak{p}}^{\tau}(L)$, for all $L \in \mathcal{K}$. Also note that at least in the case $\tau = (1, 1)$, the Kochen operator $\gamma_{\mathfrak{p}, \tau}^{\tau}$ is universal in the sense that every such δ is in fact a rational function in $\gamma_{\mathfrak{p}, \tau}^{\tau}$, see [22, Cor. 7.12].

4 | THE (\mathfrak{p}, τ) -PYTHAGORAS NUMBER OF NUMBER FIELDS

Introduced by Poonen ([21]), and subsequently used and developed by others including Koenigsmann ([14]) and the second author ([7]), the following diophantine predicates behave well in local fields, and satisfy a strong local-global principle. They are defined from central simple algebras. For further details about central simple algebras, the Brauer group, and associated local-global principles, see [19, Sect. 6.3].

Let A be a central simple algebra of prime degree ℓ over a field F . Following [7, Sect. 2], we let

$$S_A(F) := \left\{ \text{Trd}(x) \mid x \in A, \text{Nrd}(x) = 1 \right\} \subseteq F,$$

where Trd and Nrd are the reduced norm and reduced trace, see [12, Construction 2.6.1] for details. We also define

$$T_A(F) := \begin{cases} S_A(F) & \text{if } \ell > 2, \\ S_A(F) - S_A(F) & \text{if } \ell = 2. \end{cases}$$

If A is a central simple algebra over F and E/F is any extension, we view $A_E := A \otimes_F E$ as a central simple algebra over E and write $S_A(E) := S_{A_E}(E)$ and $T_A(E) := T_{A_E}(E)$.

Lemma 4.1. *Both S_A and T_A are 1-dimensional diophantine families over F .*

Proof. This is shown in [7, Lem. 2.12] and the subsequent discussion. □

Recall that A is *split* if it is isomorphic to a matrix algebra over F , and A splits over E if A_E is split. The behaviour of S_A and T_A in a completion F of a number field L is determined by whether or not A splits over F , and the behaviour of S_A and T_A in L is controlled by a local-global principle, which leads to the following:

Proposition 4.2 ([7, Prop. 2.9]). *Let L be a number field and A a central simple algebra over L of prime degree ℓ which splits over all real completions of L . Then*

$$T_A(L) = \bigcap_{\mathfrak{p}} \mathcal{O}_{\mathfrak{p}},$$

where the intersection is over the finitely many finite primes \mathfrak{p} of L such that A does not split over $L_{\mathfrak{p}}$.

Proposition 4.3 (see [7, Prop. 2.6]). *Let F be a non-archimedean local field of characteristic zero and let A be a central simple algebra over F of prime degree ℓ . If A is non-split then $T_A(F) = \mathcal{O}_F$.*

Note that [7, Prop. 2.6] is stated for central division algebras of prime degree, but a non-split central simple algebra of prime degree is a division algebra.

Recall that above we fixed a number field K , a finite place \mathfrak{p} of K , and a pair $\tau = (e, f) \in \mathbb{N}^2$. Given this data (K, \mathfrak{p}, τ) , we now describe a choice of algebras A, B over K .

Proposition 4.4. *For every prime number ℓ there exist central simple algebras A, B of degree ℓ over K such that*

1. *neither of them splits over $K_{\mathfrak{p}}$,*
2. *for every finite place $\mathfrak{q} \neq \mathfrak{p}$ of K , at least one of them splits over $K_{\mathfrak{q}}$,*
3. *for every infinite place \mathfrak{q} of K , both of them split over $K_{\mathfrak{q}}$.*

Proof. The Brauer equivalence classes $[A]$ of central simple algebras A over a field F form the Brauer group $\text{Br}(F)$ of F , see [19, (6.3.2) Def.]. For an extension F/K , there is a group homomorphism $\text{Br}(K) \rightarrow \text{Br}(F)$ given by $[A] \mapsto [A_F]$. Moreover,

the local Hasse invariant is an isomorphism

$$\text{inv}_{K_q} : \text{Br}(K_q) \rightarrow \begin{cases} \mathbb{Q}/\mathbb{Z} & \text{if } q \text{ is finite,} \\ \frac{1}{2}\mathbb{Z}/\mathbb{Z} & \text{if } q \text{ is infinite and } K_q \cong \mathbb{R}, \\ 0 & \text{if } q \text{ is infinite and } K_q \cong \mathbb{C}, \end{cases} \quad (4.1)$$

and so A splits over K_q if and only if $\text{inv}_{K_q}([A]) = 0$. There will be no ambiguity if we write $\text{inv}_{K_q}([A]) = \text{inv}_{K_q}([A_{K_q}])$. Note that each of the local Hasse invariants inv_{K_q} takes its values in \mathbb{Q}/\mathbb{Z} .

The Albert–Brauer–Hasse–Noether Theorem ([19, (8.1.17) Thm.]) gives the exact sequence

$$0 \rightarrow \text{Br}(K) \longrightarrow \bigoplus_{q \in \mathbb{S}(K)} \text{Br}(K_q) \xrightarrow{\text{inv}_K} \mathbb{Q}/\mathbb{Z} \rightarrow 0, \quad (4.2)$$

where $\mathbb{S}(K)$ is the set of (finite and infinite) places of K , and inv_K is the sum of the local invariant maps inv_{K_q} .

Fix two distinct finite places $q_1, q_2 \neq p$ of K . We define two sequences $(a_q)_{q \in \mathbb{S}(K)}$ and $(b_q)_{q \in \mathbb{S}(K)}$ of rational numbers, indexed by the places of K , by

- $a_p = b_p = \ell^{-1}$,
- $a_{q_1} = (\ell - 1)\ell^{-1}$ and $b_{q_1} = 0$,
- $a_{q_2} = 0$ and $b_{q_2} = (\ell - 1)\ell^{-1}$,
- $a_q = b_q = 0$, for every other place q .

Note that only finitely many of the elements of these sequences are nonzero. Thus, by applying the inverses of the local Hasse invariants from (a), the sequences $(a_q)_q$ and $(b_q)_q$ correspond to elements of the direct sum $\bigoplus_q \text{Br}(K_q)$. We also note the sums

$$\sum_{q \in \mathbb{S}(K)} a_q = \sum_{q \in \mathbb{S}(K)} b_q = 0 \quad \text{in } \mathbb{Q}/\mathbb{Z}.$$

By the exactness of the short exact sequence (4.2), we get (unique) equivalence classes $[A]$ and $[B]$ in $\text{Br}(K)$ such that $\text{inv}_{K_q}([A]) = a_q + \mathbb{Z}$ and $\text{inv}_{K_q}([B]) = b_q + \mathbb{Z}$, for all $q \in \mathbb{S}(K)$. Thus both $[A]$ and $[B]$ are of period ℓ . As K is a number field, this implies that they are also of index ℓ ([23, 32.19]), which means that if A and B denote the unique division algebras in $[A]$ respectively $[B]$, then these are of degree ℓ . \square

Proposition 4.5. *Let ℓ be a prime number with $\ell > ef$. If A and B are algebras as in Proposition 4.4, then*

(i) *for all finite extensions E/K_p of relative type at most τ ,*

$$T_A(E) + T_B(E) = \mathcal{O}_E;$$

(ii) *and for all number fields L/K ,*

$$T_A(L) + T_B(L) \supseteq \bigcap_{\mathfrak{p} \in S_p^*(L)} \mathcal{O}_{\mathfrak{p}}.$$

Proof. First, suppose that E/K_p is a finite extension of relative type at most τ . Thus $[E : K_p] \leq ef < \ell$, so since A and B do not split over K_p , they also do not split over E by [12, Cor. 4.5.9]. Therefore we may apply Proposition 4.3 to obtain

$$T_A(E) + T_B(E) = \mathcal{O}_E + \mathcal{O}_E = \mathcal{O}_E.$$

Next, let L/K be any number field and let \mathfrak{Q} be a prime of L which lies over a prime q of K . If $q \neq p$, then at least one of A and B splits over K_q and therefore also over the completion $L_{\mathfrak{Q}}$ by construction. Hence

$$T_A(L) + T_B(L) = \bigcap_{\substack{\mathfrak{Q} \in S(L) \\ A_{L_{\mathfrak{Q}}} \text{ not split}}} \mathcal{O}_{\mathfrak{Q}} + \bigcap_{\substack{\mathfrak{Q} \in S(L) \\ B_{L_{\mathfrak{Q}}} \text{ not split}}} \mathcal{O}_{\mathfrak{Q}} = \bigcap_{\substack{\mathfrak{Q} \in S(L) \\ A_{L_{\mathfrak{Q}}} \text{ and } B_{L_{\mathfrak{Q}}} \text{ not split}}} \mathcal{O}_{\mathfrak{Q}} \supseteq \bigcap_{\mathfrak{p} \in S_p^*(L)} \mathcal{O}_{\mathfrak{p}},$$

where the first equality is Proposition 4.2 and the second equality follows from weak approximation (see e.g. [9, 1.1.3]). \square

As before, fix a uniformizer $t_{\mathfrak{p}} \in K$ of \mathfrak{p} . For central simple algebras A, B over K and an extension F/K we define $D_{\mathfrak{p}, t_{\mathfrak{p}}, A, B}^{\tau}(F)$ as

$$\left\{ \frac{x}{1 + t_{\mathfrak{p}} w^{e+1} y} \mid x, y \in T_A(F) + T_B(F), w \in \gamma_{\mathfrak{p}, t_{\mathfrak{p}}}^{\tau}(F), 1 + t_{\mathfrak{p}} w^{e+1} y \neq 0 \right\}.$$

Lemma 4.6. $D_{\mathfrak{p}, t_{\mathfrak{p}}, A, B}^{\tau}$ is a 1-dimensional diophantine family over K .

Proof. We have seen in Lemma 4.1 that T_A and T_B are 1-dimensional diophantine families over K . The claim follows by applying Lemma 3.5 to the 5-dimensional diophantine family $T_A \times T_B \times T_A \times T_B \times \gamma_{\mathfrak{p}, t_{\mathfrak{p}}}^{\tau}$ over K (Lemma 3.4) and the rational function $(X_1 + X_2)(1 + t_{\mathfrak{p}} X_5^{e+1}(X_3 + X_4))^{-1}$. \square

Proposition 4.7. If A, B are K -algebras as in Proposition 4.4, then

$$D_{\mathfrak{p}, t_{\mathfrak{p}}, A, B}^{\tau}(E) \subseteq \mathcal{O}_E$$

for every finite extension $E/K_{\mathfrak{p}}$ of relative type at most τ .

Proof. By Proposition 4.5(i), we have $T_A(E) + T_B(E) = \mathcal{O}_E$. Since also $\gamma_{\mathfrak{p}, t_{\mathfrak{p}}}^{\tau}(E) \subseteq \mathcal{O}_E$ and $1 + t_{\mathfrak{p}} \mathcal{O}_E \subseteq \mathcal{O}_E^{\times}$, we have $D_{\mathfrak{p}, t_{\mathfrak{p}}, A, B}^{\tau}(E) \subseteq \mathcal{O}_E$, as required. \square

Proposition 4.8. If A, B are K -algebras as in Proposition 4.4, then

$$D_{\mathfrak{p}, t_{\mathfrak{p}}, A, B}^{\tau}(L) = R_{\mathfrak{p}}^{\tau}(L)$$

for every number field L containing K .

Proof. By Proposition 4.7, $D_{\mathfrak{p}, t_{\mathfrak{p}}, A, B}^{\tau}(L_{\mathfrak{P}}) \subseteq \mathcal{O}_{L_{\mathfrak{P}}}$ for every $\mathfrak{P} \in S_{\mathfrak{p}}^{\tau}(L)$, hence

$$D_{\mathfrak{p}, t_{\mathfrak{p}}, A, B}^{\tau}(L) \subseteq \bigcap_{\mathfrak{P} \in S_{\mathfrak{p}}^{\tau}(L)} \mathcal{O}_{L_{\mathfrak{P}}} \cap L = \bigcap_{\mathfrak{P} \in S_{\mathfrak{p}}^{\tau}(L)} \mathcal{O}_{\mathfrak{P}} = R_{\mathfrak{p}}^{\tau}(L).$$

To show the other inclusion, let $r \in R_{\mathfrak{p}}^{\tau}(L)$. Since L/K is finite, the set $S_{\mathfrak{p}}^*(L)$ of primes of L over \mathfrak{p} is finite. Write $\mathfrak{P}_1, \dots, \mathfrak{P}_k \in S_{\mathfrak{p}}^{\tau}(L)$ for the primes over \mathfrak{p} of relative type $\leq \tau$, and $\mathfrak{Q}_1, \dots, \mathfrak{Q}_l$ for the primes over \mathfrak{p} not of relative type $\leq \tau$. For each $i \in \{1, \dots, l\}$, by Lemma 2.12 there exists z_i such that

$$v_{\mathfrak{Q}_i}(\gamma_{\mathfrak{p}, t_{\mathfrak{p}}}^{\tau}(z_i)) \leq -\frac{1}{e+1} v_{\mathfrak{Q}_i}(t_{\mathfrak{p}}),$$

i.e. $v_{\mathfrak{Q}_i} \left(\left(t_{\mathfrak{p}} \gamma_{\mathfrak{p}, t_{\mathfrak{p}}}^{\tau}(z_i)^{e+1} \right)^{-1} \right) \geq 0$. By weak approximation and continuity of rational functions, there exists $z \in L$ such that $v_{\mathfrak{Q}_i} \left(\left(t_{\mathfrak{p}} \gamma_{\mathfrak{p}, t_{\mathfrak{p}}}^{\tau}(z)^{e+1} \right)^{-1} \right) \geq 0$ for each $i \in \{1, \dots, l\}$. By another application of weak approximation there exists $y \in L$ such that

$$v_{\mathfrak{Q}_i} \left(\left(t_{\mathfrak{p}} \gamma_{\mathfrak{p}, t_{\mathfrak{p}}}^{\tau}(z)^{e+1} \right)^{-1} + y \right) \geq \max \left\{ 0, -v_{\mathfrak{Q}_i} \left(r t_{\mathfrak{p}} \gamma_{\mathfrak{p}, t_{\mathfrak{p}}}^{\tau}(z)^{e+1} \right) \right\}, \quad i = 1, \dots, l,$$

$$v_{\mathfrak{P}_i}(y) \geq 0, \quad i = 1, \dots, k.$$

In particular, $y \in \bigcap_{\mathfrak{P} \in S_{\mathfrak{p}}^*(L)} \mathcal{O}_{\mathfrak{P}}$ and $x := r \left(1 + t_{\mathfrak{p}} \gamma_{\mathfrak{p}, t_{\mathfrak{p}}}^{\tau}(z)^{e+1} y \right)$ satisfies $v_{\mathfrak{Q}_i}(x) \geq 0$ for each $i \in \{1, \dots, l\}$. As $\mathfrak{P}_i \in S_{\mathfrak{p}}^{\tau}(L)$, we have $r, t_{\mathfrak{p}}, \gamma_{\mathfrak{p}, t_{\mathfrak{p}}}^{\tau}(z), y \in \mathcal{O}_{\mathfrak{P}_i}$, hence $v_{\mathfrak{P}_i}(x) \geq 0$ for all $i \in \{1, \dots, k\}$. Thus $x \in \bigcap_{\mathfrak{P} \in S_{\mathfrak{p}}^*(L)} \mathcal{O}_{\mathfrak{P}}$. As

$$\bigcap_{\mathfrak{P} \in S_{\mathfrak{p}}^*(L)} \mathcal{O}_{\mathfrak{P}} \subseteq T_A(L) + T_B(L)$$

by Proposition 4.5(ii), we get that

$$r = x \left(1 + t_{\mathfrak{p}} \gamma_{\mathfrak{p}, t_{\mathfrak{p}}}^{\tau}(z)^{e+1} y \right)^{-1} \in D_{\mathfrak{p}, t_{\mathfrak{p}}, A, B}^{\tau}(L),$$

as required. \square

Theorem 4.9. *For every finite place \mathfrak{p} of a number field K and every $\tau \in \mathbb{N}^2$, there exists $N \in \mathbb{N}$ such that $\pi_{\mathfrak{p}}^{\tau}(L) \leq N$ for every number field L containing K .*

Proof. We choose algebras A and B over K according to Proposition 4.4, and we apply Proposition 3.10 to the class \mathcal{K} of finite extensions L/K and the diophantine family $D = D_{\mathfrak{p}, t_{\mathfrak{p}}, A, B}^{\tau}$, where the two assumptions of Proposition 3.10 are verified in Proposition 4.8 and Proposition 4.7, respectively. \square

Remark 4.10. Given an arbitrary field $F \supseteq K$ there is no obvious relation between $\pi_{\mathfrak{p}}^{\tau}(F)$ and $\pi_{\mathfrak{p}}^{\tau'}(F)$ for $\tau \neq \tau'$. For example if $\tau \leq \tau'$ then we have $R_{\mathfrak{p}}^{\tau}(F) \supseteq R_{\mathfrak{p}}^{\tau'}(F)$, but also $\gamma_{\mathfrak{p}}^{\tau} \neq \gamma_{\mathfrak{p}}^{\tau'}$. Likewise, there is no reason to suspect that the bounds N in Theorem 4.9 should be related for different choices of τ .

5 | THE (\mathfrak{p}, τ) -PYTHAGORAS NUMBER IN FINITE EXTENSIONS

The growth of the classical Pythagoras number is bounded in finite extensions E/F by

$$\pi(E) \leq [E : F] \cdot \pi(F),$$

see [20, Ch. 7, Prop. 1.13]. We now combine ideas from the proof of Theorem 4.9 with techniques for p -valuations on general fields to prove an (inexplicit) analogue of this for the (\mathfrak{p}, τ) -Pythagoras number.

As before fix K , \mathfrak{p} and $\tau = (e, f)$ and let F/K be an extension. We equip $S_{\mathfrak{p}}^{\tau}(F)$ with the *constructible topology*, which by definition has a basis consisting of the sets

$$S_{\mathfrak{p}}^{\tau}(F; a) := \{ \mathfrak{P} \in S_{\mathfrak{p}}^{\tau}(F) \mid v_{\mathfrak{P}}(a) \geq 0 \}, \quad a \in F,$$

and their complements. In [1], we studied approximation theorems for spaces of localities, i.e. valuations, orderings, and absolute values, on a given field. We now deduce an approximation theorem in the setting of the space $S_{\mathfrak{p}}^{\tau}(F)$.

Theorem 5.1. *Let $S_1, \dots, S_n \subseteq S_{\mathfrak{p}}^{\tau}(F)$ be disjoint and closed, let $x_1, \dots, x_n \in F$, and let $z_1, \dots, z_n \in F^{\times}$. Assume that, for any $\mathfrak{P}_i \in S_i$ and $\mathfrak{P}_j \in S_j$, if the valuation w is the finest common coarsening of $v_{\mathfrak{P}_i}$ and $v_{\mathfrak{P}_j}$, then $w(x_i - x_j) \geq w(z_i) = w(z_j)$. Then there exists $x \in F$ with*

$$v_{\mathfrak{Q}}(x - x_i) > v_{\mathfrak{Q}}(z_i) \text{ for all } \mathfrak{Q} \in S_i, \text{ for } i = 1, \dots, n.$$

Proof. Corollary 5.5 of [1] is a similar statement in which $S_{\mathfrak{p}}^{\tau}(F)$ is replaced by a space $S_{\pi}^e(F)$, for $\pi \in F^{\times}$ and $e \in \mathbb{N}$. By definition (see [1, Example 2.4]), $S_{\pi}^e(F)$ is the space of equivalence classes of valuations v on F with value group Γ_v , which has \mathbb{Z} as a convex subgroup and $0 < v(\pi) \leq e$. We note that $S_{\mathfrak{p}}^{\tau}(F) \subseteq S_{t_{\mathfrak{p}}}^e(F)$, and if we equip $S_{t_{\mathfrak{p}}}^e(F)$ with its own constructible topology (see [1, Sect. 2]) then $S_{\mathfrak{p}}^{\tau}(F)$ is a closed subspace: By [22, Lem. 6.2], $S_{\mathfrak{p}}^{\tau}(F)$ is the intersection over all sets $\{v \in S_{t_{\mathfrak{p}}}^e(F) : v(a) \geq 0\}$ for $a \in \mathcal{O}_{\mathfrak{p}} \cup \gamma_{\mathfrak{p}, t_{\mathfrak{p}}}^{\tau}(F)$. Therefore, each S_i is also a closed subset of $S_{t_{\mathfrak{p}}}^e(F)$ and so we may obtain the required element $x \in F$ by an application of [1, Cor. 5.5]. \square

Lemma 5.2. *Let $\tau \leq \tau' \in \mathbb{N}^2$. There is a rational function $\omega_{\tau, \tau'} \in \mathbb{Q}(t_{\mathfrak{p}})(X)$ such that $v_{\mathfrak{P}}(\omega_{\tau, \tau'}(x)) > 0$ for all $x \in F$ and $\mathfrak{P} \in S_{\mathfrak{p}}^{\tau'}(F)$, and moreover $v_{\mathfrak{P}}(\omega_{\tau, \tau'}(x)) = 1$ if $v_{\mathfrak{P}}(x) = 1$ and \mathfrak{P} is of exact relative type τ over \mathfrak{p} .*

Proof. Write $\tau' = (e', f')$. By Dirichlet's theorem on primes in arithmetic progressions there exists $k \in \mathbb{N}$ such that $\ell := 1 + ke$ is a prime number and $\ell > e'$. Let $\beta(X) = t_{\mathfrak{p}}^{-k} X^{\ell}$. For every $\mathfrak{P} \in S_{\mathfrak{p}}^{\tau'}(F)$ and $x \in F$ we have $v_{\mathfrak{P}}(\beta(x)) = \ell v_{\mathfrak{P}}(x) - kv_{\mathfrak{P}}(t_{\mathfrak{p}})$, which is non-zero (since $\ell > k$ and $\ell > e' \geq v_{\mathfrak{P}}(t_{\mathfrak{p}})$ imply $\ell \nmid kv_{\mathfrak{P}}(t_{\mathfrak{p}})$), and equals 1 if $v_{\mathfrak{P}}(x) = 1$ and $v_{\mathfrak{P}}(t_{\mathfrak{p}}) = e$. Thus $\omega_{\tau, \tau'}(X) = (\beta(X) + \beta(X)^{-1})^{-1}$ satisfies the claim. \square

Lemma 5.3. *There is a rational function $\rho_\tau \in \mathbb{Q}(X)$ such that for all $\mathfrak{P} \in S_p^\tau(F)$ and all $x \in F$ we have*

$$v_{\mathfrak{P}}(\rho_\tau(x)) \begin{cases} = 0, & \text{if } v_{\mathfrak{P}}(x) = 0, \\ > 0, & \text{if } v_{\mathfrak{P}}(x) \neq 0, \end{cases}$$

and if $v_{\mathfrak{P}}(x) = 0$ then $\text{res}_{\mathfrak{P}}(\rho_\tau(x)) = \text{res}_{\mathfrak{P}}(x)$.

Proof. Write $\rho_\tau(X) = X(X^{q^f} - X + 1)^{-1}$. Let $\mathfrak{P} \in S_p^\tau(F)$ and let $x \in F$. If $v_{\mathfrak{P}}(x) < 0$ then $v_{\mathfrak{P}}(x^{q^f} - x + 1) = q^f v_{\mathfrak{P}}(x) < 0$, and so $v_{\mathfrak{P}}(\rho_\tau(x)) = (1 - q^f)v_{\mathfrak{P}}(x) > 0$. On the other hand, if $v_{\mathfrak{P}}(x) > 0$ then $v_{\mathfrak{P}}(x^{q^f} - x + 1) = 0$, so $v_{\mathfrak{P}}(\rho_\tau(x)) = v_{\mathfrak{P}}(x) > 0$. Finally, if $v_{\mathfrak{P}}(x) = 0$ then

$$\text{res}_{\mathfrak{P}}(x^{q^f} - x + 1) = \text{res}_{\mathfrak{P}}(x)^{q^f} - \text{res}_{\mathfrak{P}}(x) + 1 = 1 \neq 0,$$

and in particular $v_{\mathfrak{P}}(x^{q^f} - x + 1) = 0$. Therefore $v_{\mathfrak{P}}(\rho_\tau(x)) = 0$ and $\text{res}_{\mathfrak{P}}(\rho_\tau(x)) = \text{res}_{\mathfrak{P}}(x)$. \square

Proposition 5.4. *Let $\tau \leq \tau' = (e', f')$ and let S_0 denote an open-closed subset of $S_p^{\tau'}(F)$ such that $S_p^\tau(F) \subseteq S_0$. There exists $y \in F$ such that*

$$v_{\mathfrak{P}}(\gamma_{p, t_p}^\tau(y)) \begin{cases} \in [0, e'eq^f], & \text{if } \mathfrak{P} \in S_0, \\ < 0, & \text{if } \mathfrak{P} \in S_p^{\tau'}(F) \setminus S_0. \end{cases}$$

Proof. For each $\mathfrak{P} \in S_p^{\tau'}(F) \setminus S_0$, we choose $y_{\mathfrak{P}} \in F$ as follows. First, if the relative type of \mathfrak{P} is exactly $\tau'' = (e'', f'')$ with $e'' > e$, then let $t_{\mathfrak{P}}$ be a uniformizer of $v_{\mathfrak{P}}$ and set $y_{\mathfrak{P}} = \omega_{\tau'', \tau'}(t_{\mathfrak{P}})$. By Lemma 5.2, $v_{\mathfrak{P}}(y_{\mathfrak{P}}) = 1$; and by Lemma 2.12, $v_{\mathfrak{P}}(\gamma_{p, t_p}^\tau(y_{\mathfrak{P}})) < 0$. Also, for all $\mathfrak{Q} \in S_p^{\tau'}(F)$ we have $v_{\mathfrak{Q}}(y_{\mathfrak{P}}) > 0$. In particular, $y_{\mathfrak{P}} \in R_p^{\tau'}(F)$.

On the other hand, if the relative type of \mathfrak{P} is exactly $\tau'' = (e'', f'')$ with $f'' \nmid f$, then let $a_{\mathfrak{P}}$ with $v_{\mathfrak{P}}(a_{\mathfrak{P}}) = 0$ and $\text{res}_{\mathfrak{P}}(a_{\mathfrak{P}})$ a generator of $Fv_{\mathfrak{P}}$, and set $y_{\mathfrak{P}} = \rho_{\tau'}(a_{\mathfrak{P}})$. By Lemma 5.3, $v_{\mathfrak{P}}(y_{\mathfrak{P}}) = 0$ and $\text{res}_{\mathfrak{P}}(y_{\mathfrak{P}})$ is a generator of $Fv_{\mathfrak{P}}$. By Lemma 2.12, we have $v_{\mathfrak{P}}(\gamma_{p, t_p}^\tau(y_{\mathfrak{P}})) < 0$. Also, for all $\mathfrak{Q} \in S_p^{\tau'}(F)$ we have $v_{\mathfrak{Q}}(y_{\mathfrak{P}}) \geq 0$, i.e. $y_{\mathfrak{P}} \in R_p^{\tau'}(F)$.

In either case, we have chosen $y_{\mathfrak{P}} \in R_p^{\tau'}(F)$ such that $v_{\mathfrak{P}}(\gamma_{p, t_p}^\tau(y_{\mathfrak{P}})) < 0$. Next we make use of the compactness of $S_p^{\tau'}(F)$. For $y \in F$, we let

$$S_y = \left\{ \mathfrak{P} \in S_p^{\tau'}(F) \mid v_{\mathfrak{P}}(\gamma_{p, t_p}^\tau(y)) < 0 \right\}.$$

Each S_y is an open-closed subset of $S_p^{\tau'}(F)$. By our choice of the elements $y_{\mathfrak{P}}$, the family

$$\{S_{y_{\mathfrak{P}}} \setminus S_0 : \mathfrak{P} \in S_p^{\tau'}(F) \setminus S_0\}$$

is an open covering of $S_p^{\tau'}(F) \setminus S_0$. So by compactness there exist $\mathfrak{P}_1, \dots, \mathfrak{P}_n \in S_p^{\tau'}(F) \setminus S_0$ such that with $S'_i := S_{y_{\mathfrak{P}_i}}$, we have

$$S_p^{\tau'}(F) = S_0 \cup S'_1 \cup \dots \cup S'_n.$$

Choose open-closed sets $S_1 \subseteq S'_1, \dots, S_n \subseteq S'_n$ such that

$$S_p^{\tau'}(F) = S_0 \sqcup S_1 \sqcup \dots \sqcup S_n$$

is a partition. We seek to apply Theorem 5.1 to the sets S_0, S_1, \dots, S_n , the elements $x_0 = t_p^{-1}$, $x_1 = y_{\mathfrak{P}_1}, \dots, x_n = y_{\mathfrak{P}_n}$ and $z_0 = t_p, \dots, z_n = t_p$. To verify that the hypothesis of the theorem holds, we argue as follows: let w be any valuation on F that is a common coarsening of valuations $v_{\mathfrak{P}}$ and $v_{\mathfrak{Q}}$ corresponding to primes $\mathfrak{P} \in S_i$ and $\mathfrak{Q} \in S_j$, for $i \neq j$. Note that w is a proper coarsening of these valuations since S_i and S_j are disjoint and $v_{\mathfrak{P}}, v_{\mathfrak{Q}}$ are incomparable. Then $w(z_i) = w(z_j) = 0$ and $w(x_i - x_j) \geq 0$. Therefore, by Theorem 5.1, there exists $y \in F$ such that

$$v_{\mathfrak{P}}(y - x_i) > v_{\mathfrak{P}}(t_p),$$

for each $\mathfrak{P} \in S_i$ and each i . In particular, for $\mathfrak{P} \in S_0$ we have that $v_{\mathfrak{P}}(y) = -v_{\mathfrak{P}}(t_{\mathfrak{P}}) < 0$, hence

$$v_{\mathfrak{P}}(\gamma_{\mathfrak{P},t_{\mathfrak{P}}}^{\tau}(y)) = eq^f v_{\mathfrak{P}}(t_{\mathfrak{P}}) - v_{\mathfrak{P}}(t_{\mathfrak{P}}) = (eq^f - 1)v_{\mathfrak{P}}(t_{\mathfrak{P}}) \in \{0, \dots, e'e q^f\},$$

cf. Lemma 2.11. On the other hand, for $\mathfrak{Q} \in S_i$, with $i > 0$, we get that $v_{\mathfrak{Q}}(y - y_{\mathfrak{P}_i}) > v_{\mathfrak{Q}}(t_{\mathfrak{P}})$. Since we have $v_{\mathfrak{Q}}(\gamma_{\mathfrak{P},t_{\mathfrak{P}}}^{\tau}(y_{\mathfrak{P}_i})) < 0$, then $v_{\mathfrak{Q}}(\gamma_{\mathfrak{P},t_{\mathfrak{P}}}^{\tau}(y)) < 0$ by Lemma 2.13. \square

Fix $n, m \in \mathbb{N}$ and let $\tau' = (e', f')$, where $e' = me$ and $f' = m!f$. Let \mathcal{E} be the class of fields E which contain some F/K with $[E : F] = m$ and $\pi_{\mathfrak{P}}^{\tau}(F) = n$. We adapt the arguments of Section 4 in order to show that $\pi_{\mathfrak{P}}^{\tau'}(E)$ is bounded by a function of m, n . We let

$$D_{\mathfrak{p},m,n}^{\tau,(1)}(F) := \{x \in F \mid \exists a_0, \dots, a_{m-1} \in R_{\mathfrak{p},n}^{\tau}(F) : x^m + a_{m-1}x^{m-1} + \dots + a_0 = 0\},$$

and

$$D_{\mathfrak{p},m,n}^{\tau,(2)}(F) := \left\{ \frac{a}{1 + t_{\mathfrak{P}}\gamma_{\mathfrak{P},t_{\mathfrak{P}}}^{\tau}(y)^{e'}b} \mid a, b \in D_{\mathfrak{p},m,n}^{\tau,(1)}(F), y \in F, \gamma_{\mathfrak{P},t_{\mathfrak{P}}}^{\tau}(y) \neq \infty, 1 + t_{\mathfrak{P}}\gamma_{\mathfrak{P},t_{\mathfrak{P}}}^{\tau}(y)^{e'}b \neq 0 \right\}.$$

Lemma 5.5. *Both $D_{\mathfrak{p},m,n}^{\tau,(1)}$ and $D_{\mathfrak{p},m,n}^{\tau,(2)}$ are 1-dimensional diophantine families over K .*

Proof. This is very similar to Lemma 4.6. This time we use the fact that $R_{\mathfrak{p},n}^{\tau}$ is a 1-dimensional diophantine family over K , as seen in Example 3.8. From this it immediately follows that $D_{\mathfrak{p},m,n}^{\tau,(1)}$ is a 1-dimensional diophantine family over K . To see that $D_{\mathfrak{p},m,n}^{\tau,(2)}$ is a 1-dimensional diophantine family over K we now apply Lemma 3.5 to the 3-dimensional diophantine family $D_{\mathfrak{p},m,n}^{\tau,(1)} \times D_{\mathfrak{p},m,n}^{\tau,(1)} \times \gamma_{\mathfrak{P},t_{\mathfrak{P}}}^{\tau}$ and the rational function $X_1(1 + t_{\mathfrak{P}}X_3^{e'}X_2)^{-1}$. \square

Proposition 5.6. *For every $E \supseteq K$ we have $D_{\mathfrak{p},m,n}^{\tau,(2)}(E) \subseteq R_{\mathfrak{P}}^{\tau}(E)$.*

Proof. Since $R_{\mathfrak{P}}^{\tau}(E)$ is integrally closed in E and $R_{\mathfrak{p},n}^{\tau}(E) \subseteq R_{\mathfrak{P}}^{\tau}(E)$, we have $D_{\mathfrak{p},m,n}^{\tau,(1)}(E) \subseteq R_{\mathfrak{P}}^{\tau}(E)$. Let $\mathfrak{P} \in S_{\mathfrak{P}}^{\tau}(E)$. Then $v_{\mathfrak{P}}(t_{\mathfrak{P}}) > 0$. Furthermore, for $y \in E$ and $b \in R_{\mathfrak{P}}^{\tau}(E)$, we have $v_{\mathfrak{P}}(\gamma_{\mathfrak{P},t_{\mathfrak{P}}}^{\tau}(y)^{e'}b) \geq 0$, hence $v_{\mathfrak{P}}(1 + t_{\mathfrak{P}}\gamma_{\mathfrak{P},t_{\mathfrak{P}}}^{\tau}(y)^{e'}b) = 0$. Therefore elements of the form $a(1 + t_{\mathfrak{P}}\gamma_{\mathfrak{P},t_{\mathfrak{P}}}^{\tau}(y)^{e'}b)^{-1}$ are contained in $R_{\mathfrak{P}}^{\tau}(E)$, where $a, b \in D_{\mathfrak{p},m,n}^{\tau,(1)}(E)$ and $y \in E$. This establishes $D_{\mathfrak{p},m,n}^{\tau,(2)}(E) \subseteq R_{\mathfrak{P}}^{\tau}(E)$. \square

Lemma 5.7. *For every $E \in \mathcal{E}$ we have $R_{\mathfrak{P}}^{\tau'}(E) \subseteq D_{\mathfrak{p},m,n}^{\tau,(1)}(E)$.*

Proof. Choose F such that $[E : F] = m$ and $\pi_{\mathfrak{P}}^{\tau}(F) = n$, although the choice of F will not matter. Let S be the set of primes of E (of arbitrary type) lying over elements of $S_{\mathfrak{P}}^{\tau}(F)$. By our choice of τ' , we have $S \subseteq S_{\mathfrak{P}}^{\tau'}(E)$. If we denote by A the integral closure of $R_{\mathfrak{P}}^{\tau}(F)$ in E , then A is the holomorphy ring corresponding to S and we have

$$R_{\mathfrak{P}}^{\tau'}(E) \subseteq A \subseteq R_{\mathfrak{P}}^{\tau}(E).$$

Since $\pi_{\mathfrak{P}}^{\tau}(F) = n$, we have $R_{\mathfrak{P}}^{\tau}(F) = R_{\mathfrak{p},n}^{\tau}(F)$; and trivially $R_{\mathfrak{p},n}^{\tau}(F) \subseteq R_{\mathfrak{p},n}^{\tau}(E)$. As the degree of the extension E/F is m , $D_{\mathfrak{p},m,n}^{\tau,(1)}(E)$ contains the integral closure of $R_{\mathfrak{P}}^{\tau}(F)$ in E , which is A . In particular $R_{\mathfrak{P}}^{\tau'}(E) \subseteq D_{\mathfrak{p},m,n}^{\tau,(1)}(E)$. \square

Proposition 5.8. *For every $E \in \mathcal{E}$ we have $D_{\mathfrak{p},m,n}^{\tau,(2)}(E) = R_{\mathfrak{P}}^{\tau}(E)$.*

Proof. In view of Proposition 5.6, it only remains to show that $R_{\mathfrak{P}}^{\tau}(E) \subseteq D_{\mathfrak{p},m,n}^{\tau,(2)}(E)$. Let $x \in R_{\mathfrak{P}}^{\tau}(E)$. In fact, we aim to find $b \in R_{\mathfrak{P}}^{\tau'}(E)$ and $y \in E$ with

$$x(1 + t_{\mathfrak{P}}\gamma_{\mathfrak{P},t_{\mathfrak{P}}}^{\tau}(y)^{e'}b) \in R_{\mathfrak{P}}^{\tau'}(E),$$

which we will do by applying Theorem 5.1. As $R_p^{\tau'}(E) \subseteq D_{p,m,n}^{\tau,(1)}$ by Lemma 5.7, this will show that $x \in D_{p,m,n}^{\tau,(2)}(E)$. We define the sets

$$S_0 := \{\mathfrak{P} \in S_p^{\tau'}(E) \mid v_{\mathfrak{P}}(x) \geq 0\}$$

and

$$S_1 := S_p^{\tau'}(E) \setminus S_0.$$

Note that S_0 and S_1 are open-closed in $S_p^{\tau'}(E)$ and $S_1 \cap S_p^{\tau}(E) = \emptyset$. We find a suitable element $y \in E$ by a direct application of Proposition 5.4: we obtain $y \in E$ such that

$$v_{\mathfrak{P}}(\gamma_{p,t_p}^{\tau}(y)) \begin{cases} \in [0, e'eq^f], & \text{if } \mathfrak{P} \in S_0, \\ < 0, & \text{if } \mathfrak{P} \in S_1. \end{cases}$$

We obtain a suitable $b \in E$ by solving a more straightforward approximation problem: By Theorem 5.1, there exists $b \in R_p^{\tau'}(E)$ such that

$$v_{\mathfrak{P}}(b) \geq 0, \quad \text{if } \mathfrak{P} \in S_0,$$

$$\text{and} \quad v_{\mathfrak{P}}\left(b + t_p^{-1} \gamma_{p,t_p}^{\tau}(y)^{-e'}\right) \geq v_{\mathfrak{P}}\left(x^{-1} t_p^{-1} \gamma_{p,t_p}^{\tau}(y)^{-e'}\right), \quad \text{if } \mathfrak{P} \in S_1.$$

Indeed, if a valuation w on E coarsens $v_{\mathfrak{P}}$ and $v_{\mathfrak{Q}}$ for $\mathfrak{P} \in S_0$ and $\mathfrak{Q} \in S_1$, $v_{\mathfrak{P}}(x) \geq 0$ and $v_{\mathfrak{Q}}(x) < 0$ imply that $w(x) = 0$, and $v_{\mathfrak{P}}(\gamma_{p,t_p}^{\tau}(y)) \in [0, e'eq^f]$ implies that $w(\gamma_{p,t_p}^{\tau}(y)) = 0$. Therefore also $w(t_p \gamma_{p,t_p}^{\tau}(y)^{e'}) = 0$ and $w(x t_p \gamma_{p,t_p}^{\tau}(y)^{e'}) = 0$. In particular, the hypothesis of the theorem is satisfied, and the $b \in E$ so obtained lies in $R_p^{\tau'}(E)$.

For $\mathfrak{P} \in S_0$, we have $v_{\mathfrak{P}}(t_p^{-1} \gamma_{p,t_p}^{\tau}(y)^{-e'}) < 0$, hence

$$v_{\mathfrak{P}}\left(b + t_p^{-1} \gamma_{p,t_p}^{\tau}(y)^{-e'}\right) \begin{cases} = v_{\mathfrak{P}}\left(t_p^{-1} \gamma_{p,t_p}^{\tau}(y)^{-e'}\right), & \text{if } \mathfrak{P} \in S_0, \\ \geq v_{\mathfrak{P}}\left(x^{-1} t_p^{-1} \gamma_{p,t_p}^{\tau}(y)^{-e'}\right), & \text{if } \mathfrak{P} \in S_1, \end{cases}$$

i.e.

$$\begin{aligned} v_{\mathfrak{P}}\left(1 + t_p \gamma_{p,t_p}^{\tau}(y)^{e'} b\right) &= 0, & \text{if } \mathfrak{P} \in S_0, \\ v_{\mathfrak{P}}\left(x\left(1 + t_p \gamma_{p,t_p}^{\tau}(y)^{e'} b\right)\right) &\geq 0, & \text{if } \mathfrak{P} \in S_1. \end{aligned}$$

Since $v_{\mathfrak{P}}(x) \geq 0$ for $\mathfrak{P} \in S_0$, we obtain that $x\left(1 + t_p \gamma_{p,t_p}^{\tau}(y)^{e'} b\right) \in R_p^{\tau'}(E)$. □

Theorem 5.9. *There is a function $\alpha_p^{\tau} : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ such that*

$$\pi_p^{\tau}(E) \leq \alpha_p^{\tau}(\pi_p^{\tau}(F), [E : F]),$$

for every field extension E/F with $\pi_p^{\tau}(F) < \infty$.

Proof. Let $m, n \in \mathbb{N}$. We apply Proposition 3.10 to the class \mathcal{E} and the diophantine family $D_{p,m,n}^{\tau,(2)}$, where the two assumptions of Proposition 3.10 are verified in Proposition 5.8 and Proposition 5.6, respectively. Thus there exists N such that $\pi_p^{\tau}(E) \leq N$ for every $E \in \mathcal{E}$, so we can choose $\alpha_p^{\tau}(n, m) = N$. □

Remark 5.10. Beyond the statement of the theorem, we are unable to say much about the behaviour of the (p, τ) -Pythagoras number in finite extensions:

For example, it is known that the classical Pythagoras does not increase in finite extensions of number fields, cf. [20, Ch. 7, Example 1.4 (2) and (3)], but we don't expect this to happen for the (p, τ) -Pythagoras number.

In fact, it is known that there are finite extensions of infinite algebraic extensions of \mathbb{Q} in which the classical Pythagoras number increases, see for instance [5, Example on p. 432], and one may expect that similar examples exist for the (p, τ) -Pythagoras number. For example, if F is the closure of \mathbb{Q} under adjoining preimages of γ_p , one trivially has $R_p(F) = F = \gamma_p(F)$, hence

$\pi_p(F) = 1$. One can then deduce from a theorem of Weissauer [26, Satz 9.7] that in any proper finite extension E of F one has $R_p(E) \neq \gamma_p(E)$, and one might suspect that in fact $\pi_p(E) > 1$, although this seems not easy to prove.

6 | DIOPHANTINE HOLOMORPHY RINGS OF p -VALUATIONS

By definition, in any field F with finite (\mathfrak{p}, τ) -Pythagoras number the holomorphy ring $R_{\mathfrak{p}}^{\tau}(F)$ is a diophantine subset. In this section we generalize this observation, by showing in Corollary 6.5 that the same applies to the holomorphy rings associated to arbitrary open-closed subsets of $S_{\mathfrak{p}}^{\tau}(F)$. Theorem 6.4 is a uniform version of this fact.

As a technical tool, it turns out to be useful to extend some of the ideas from diophantine families over fields to commutative algebras which are finite-dimensional vector spaces over fields. To this end, we introduce a small piece of notation. Write $X = (X_1, \dots, X_n)$ and $Y = (Y_1, \dots, Y_m)$. For $f_1, \dots, f_r \in K[X, Y]$ and for any commutative (unital, associative) F -algebra B , we write

$$P_{f_1, \dots, f_r}(B) := \{x \in B^n \mid \exists y \in B^m : f_1(x, y) = \dots = f_r(x, y) = 0\}.$$

The following lemma is straightforward, but we include it for lack of a suitable reference.

Lemma 6.1. *Let $f_1, \dots, f_r \in K[X, Y]$ and let $l \in \mathbb{N}$. Then*

$$F^n \cap P_{f_1^l, \dots, f_r^l}(B) = \bigcap_{\mathfrak{m} \in \text{MaxSpec}(B)} (F^n \cap P_{f_1, \dots, f_r}(B/\mathfrak{m})),$$

for all extensions F/K , and all commutative F -algebras B of dimension at most l . Here F is identified with its image in B and B/\mathfrak{m} .

Proof. Let B be a commutative F -algebra which has dimension at most l as an F -vector space. As B is finite dimensional, it is Artinian, hence the Jacobson radical \mathfrak{j} of B is nilpotent ([3, Prop. 8.4]), and therefore more precisely $\mathfrak{j}^l = 0$. Then for all $s \in \{1, \dots, r\}$, all extensions F/K , all $a \in F$, $x \in F^n$, and $y \in B^m$, we have

$$\begin{aligned} f_s(x, y)^l = 0 &\iff f_s(x, y + \mathfrak{j}) = 0 \\ &\iff f_s(x, y + \mathfrak{m}) = 0, \text{ for all } \mathfrak{m} \in \text{MaxSpec}(B). \end{aligned}$$

The result now follows from the Chinese Remainder Theorem. □

Lemma 6.2. *Let $f_1, \dots, f_r \in K[X, Y]$ and let $k \in \mathbb{N}$. There exists an $(n + k)$ -dimensional diophantine family D over K such that*

$$D(F) = \left\{ (x, z) \in F^n \times F^k \mid x \in P_{f_1, \dots, f_r}(B_z) \right\},$$

for all extensions F/K , and where B_z denotes the commutative F -algebra

$$F[T] / \left(T^k + \sum_{i=0}^{k-1} z_i T^i \right).$$

Proof. In a more advanced way, this construction can be described through the Weil restriction of the affine variety cut out by the polynomials f_1, \dots, f_r , along the family of schemes described by the B_z , fibred over the parameter space \mathbb{A}^k . Alternatively, from a model-theoretic standpoint, one can prove the statement by a quantifier-free interpretation of B_z in F , uniformly in the parameter tuple z . We give an elementary description instead.

We introduce two new tuples of variables $Z = (Z_i)_{0 \leq i < k}$ and $U = (U_{i,j})_{0 \leq i < k, 1 \leq j \leq m}$. We write

$$g(Z, T) := T^k + \sum_{i=0}^{k-1} Z_i T^i \in K[Z, T]$$

and, for each $s \in \{1, \dots, r\}$, we let

$$\hat{f}_s(X, U, T) := f_s\left(X, \sum_{i=0}^{k-1} U_{i,1} T^i, \dots, \sum_{i=0}^{k-1} U_{i,m} T^i\right).$$

Choose $d \in \mathbb{N}$ to be the maximum of the degrees of the polynomials \hat{f}_s in the variable T , and introduce a new tuple of variables $W = (W_l)_{0 \leq l \leq d}$. Then, for each s , we consider the polynomial

$$\tilde{f}_s(X, Z, U, W, T) := \hat{f}_s(X, U, T) - g(Z, T) \sum_{l=0}^d W_l T^l.$$

Note that $\tilde{f}_s(x, z, u, w, T) = 0$ for some w if and only if $g(z, T)$ divides $\hat{f}_s(x, u, T)$ in $F[T]$. By taking coefficients with respect to the variable T , we obtain a family of polynomials $h_{s,l} \in K[X, Z, U, W]$, for $1 \leq s \leq r$ and $0 \leq l \leq d + k$, such that

$$\tilde{f}_s(X, Z, U, W, T) = \sum_{l=0}^{d+k} h_{s,l}(X, Z, U, W) T^l.$$

We may define the required $(n + k)$ -dimensional diophantine family D over K by writing

$$D(F) = \{(x, z) \in F^n \times F^k \mid \exists u \in F^{km}, w \in F^{d+1} : h_{s,l}(x, z, u, w) = 0 \text{ for all } s, l\},$$

for F/K . □

Lemma 6.3. *For every field extension F/K and every $a \in F$, we have*

$$S_{\mathfrak{p}}^{\tau}(F; a) = \bigcup_{\mathfrak{m} \in \text{MaxSpec}(B_a)} \text{res}_{(B_a/\mathfrak{m})/F}(S_{\mathfrak{p}}^{\tau}(B_a/\mathfrak{m})),$$

where $\text{res}_{E/F}$ denotes restriction of primes from E to F , and B_a is the commutative F -algebra

$$F[T] / \left(t_{\mathfrak{p}} a^e \left((T^{q^f} - T)^2 - 1 \right) - (T^{q^f} - T) \right).$$

Proof. Denote $\text{MaxSpec}(B_a) = \{\mathfrak{m}_1, \dots, \mathfrak{m}_r\}$ and $E_i = B_a/\mathfrak{m}_i$. Let

$$g_a = t_{\mathfrak{p}} a^e \left((T^{q^f} - T)^2 - 1 \right) - (T^{q^f} - T) \in F[T]$$

and note that g_a is closely related to $\gamma_{\mathfrak{p}, t_{\mathfrak{p}}}^{\tau}$.

First let $\mathfrak{P} \in S_{\mathfrak{p}}^{\tau}(E_i)$ for some i . If θ denotes the residue of T in E_i , we have $\gamma_{\mathfrak{p}, t_{\mathfrak{p}}}^{\tau}(\theta) \in \mathcal{O}_{\mathfrak{P}}$ and therefore $v_{\mathfrak{P}}(\theta^{q^f} - \theta) > v_{\mathfrak{P}}\left(\left(\theta^{q^f} - \theta\right)^2 - 1\right)$, so since $g_a(\theta) = 0$ we necessarily have $v_{\mathfrak{P}}(t_{\mathfrak{p}} a^e) > 0$ and therefore $v_{\mathfrak{P}}(a) \geq 0$.

Conversely, let $\mathfrak{P} \in S_{\mathfrak{p}}^{\tau}(F; a)$. Then $g_a \in \mathcal{O}_{\mathfrak{P}}[T]$ has a simple zero $T = 0$ modulo the maximal ideal of $\mathcal{O}_{\mathfrak{P}}$, which implies that there exists some i and $\mathfrak{Q} \in S_{\mathfrak{p}}^{\tau}(E_i)$ with $\mathfrak{P} = \text{res}_{E_i/F}(\mathfrak{Q})$: Indeed, if (F', v') is a henselization of $(F, v_{\mathfrak{P}})$, then $v' = v_{\mathfrak{P}'}$ for a prime \mathfrak{P}' of F' , and Hensel's lemma in the form [9, Thm. 4.1.3(4)] shows that g_a has a zero in F' , which induces an F -embedding $E_i \rightarrow F'$, and one can take $\mathfrak{Q} = \text{res}_{F'/E_i}(\mathfrak{P}')$. □

Theorem 6.4. *For every $N \in \mathbb{N}$ there exists a 2-dimensional diophantine family $D_{\mathfrak{p}, N}^{\tau}$ over K such that*

$$D_{\mathfrak{p}, N}^{\tau}(F) = \left\{ (x, a) \in F^2 \mid v_{\mathfrak{P}}(x) \geq 0 \text{ for every } \mathfrak{P} \in S_{\mathfrak{p}}^{\tau}(F; a) \right\}$$

for every extension F/K with $\pi_{\mathfrak{p}}^{\tau}(F) \leq N$.

Proof. Let $l = 2q^f$. By Theorem 5.9 there exists N' such that for all $E/F/K$ with $[E : F] \leq l$ and $\pi_{\mathfrak{p}}^{\tau}(F) \leq N$, we have $\pi_{\mathfrak{p}}^{\tau}(E) \leq N'$, and so

$$R_{\mathfrak{p}}^{\tau}(E) = R_{\mathfrak{p}, N'}^{\tau}(E). \quad (6.1)$$

By Example 3.8, $R_{\mathfrak{p}, N'}^{\tau}$ is a 1-dimensional diophantine family over K , and so we may choose polynomials $f_1, \dots, f_r \in K[X, Y_1, \dots, Y_m]$ such that

$$R_{\mathfrak{p}, N'}^{\tau}(F) = \{x \in F \mid \exists y \in F^m : f_1(x, y) = \dots = f_r(x, y) = 0\} \quad (6.2)$$

for all F/K . For each F/K with $\pi_{\mathfrak{p}}^{\tau}(F) \leq N$, and each $a \in F$, we have

$$\begin{aligned} F \cap P_{f_1^l, \dots, f_r^l}(B_a) &= \bigcap_{\mathfrak{m} \in \text{MaxSpec}(B_a)} (F \cap P_{f_1, \dots, f_r}(B_a/\mathfrak{m})) \quad \text{by Lemma 6.1,} \\ &= \bigcap_{\mathfrak{m} \in \text{MaxSpec}(B_a)} (F \cap R_{\mathfrak{p}}^{\tau}(B_a/\mathfrak{m})) \quad \text{by (6.1) and (6.2),} \\ &= \bigcap_{\mathfrak{p} \in S_{\mathfrak{p}}^{\tau}(F; a)} \mathcal{O}_{\mathfrak{p}} \quad \text{by Lemma 6.3,} \end{aligned} \quad (6.3)$$

where B_a is the l -dimensional algebra from Lemma 6.3.

By Lemma 6.2, we may define a 2-dimensional diophantine family D over K satisfying

$$D(F) = \left\{ (x, a) \in F^2 \mid x \in P_{f_1^l, \dots, f_r^l}(B_a) \right\}$$

for every extension F/K . By (6.3), for every F/K with $\pi_{\mathfrak{p}}^{\tau}(F) \leq N$ we in fact have

$$D(F) = \left\{ (x, a) \in F^2 \mid x \in \bigcap_{\mathfrak{p} \in S_{\mathfrak{p}}^{\tau}(F; a)} \mathcal{O}_{\mathfrak{p}} \right\},$$

proving the claim. □

Corollary 6.5. *If $\pi_{\mathfrak{p}}^{\tau}(F) < \infty$, then for every open-closed set $S \subseteq S_{\mathfrak{p}}^{\tau}(F)$, the holomorphy ring $\bigcap_{\mathfrak{p} \in S} \mathcal{O}_{\mathfrak{p}}$ is diophantine in F .*

Proof. As S is open-closed, it is of the form $S_{\mathfrak{p}}^{\tau}(F; a)$ for some $a \in F$, see [10, Lem. 10.4, 10.5]. Hence the claim follows from Theorem 6.4 and Lemma 3.7. □

By Example 2.5 this applies in particular to pseudo p -adically closed fields like \mathbb{Q}^{lp} , although for such fields there are in fact simpler ways of establishing Theorem 5.9.

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